

On sums of Szemerédi–Trotter sets *

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Annotation.

We prove new general results on sumsets of sets having Szemerédi–Trotter type. This family includes convex sets, sets with small multiplicative doubling, images of sets under convex/concave maps and others.

1 Introduction

Let $A = \{a_1, \dots, a_n\}$, $a_i < a_{i+1}$ be a set of real numbers. We say that A is *convex* if

$$a_{i+1} - a_i > a_i - a_{i-1}$$

for every $i = 2, \dots, n-1$. Lower bounds for sumsets/difference sets of convex sets were obtained in several papers, see [4], [1], [2], [3], [5], [16], [13], [8], [11] and others. For example, in [8] the following theorem was proved.

Theorem 1 *Let $A \subset \mathbb{R}$ be a convex set. Then*

$$|A + A| \gg |A|^{\frac{14}{9}} \log^{-\frac{2}{9}} |A|.$$

In our paper we obtain a series of results on sumsets/difference sets of rather general families of sets, including convex sets, see Theorems 11, 14 below. In particular, it allows us to refine the result above.

Theorem 2 *Let $A \subset \mathbb{R}$ be a convex set. Then*

$$|A + A| \gg |A|^{\frac{58}{37}} \log^{-\frac{20}{37}} |A|.$$

Moreover, our method gives a generalization of Theorem 1 for sumset of two *different* convex sets, see Theorem 14 below.

In [9] the authors prove a general statement on addition of a set and its image under a convex map (the first result in the direction was obtained in [3]).

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Theorem 3 *Let f be any continuous, strictly convex or concave function on the reals, and $A, C \subset \mathbb{R}$ be any finite sets such that $|A| = |C|$. Then*

$$|f(A) + C|^{10} |A + A|^9 \gg |A|^{24} \log^{-2} |A|.$$

In particular, choosing $C = f(A)$, we get

$$\max\{|f(A) + f(A)|, |A + A|\} \gg |A|^{\frac{24}{19}} \log^{-\frac{2}{19}} |A|.$$

Finally

$$|AA|^{10} |A + A|^9 \gg |A|^{24} \log^{-2} |A|.$$

We refine the result.

Theorem 4 *Let f be any continuous, strictly convex or concave function on the reals, and $A, C \subset \mathbb{R}$ be any finite sets such that $|A| = |C|$. Then*

$$|f(A) + C|^{42} |A + A|^{37} \gg |A|^{100} \log^{-20} |A|.$$

In particular, choosing $C = f(A)$, we get

$$\max\{|f(A) + f(A)|, |A + A|\} \gg |A|^{\frac{100}{79}} \log^{-\frac{20}{79}} |A|.$$

Finally

$$|AA|^{42} |A + A|^{37} \gg |A|^{100} \log^{-20} |A|.$$

Another applications can be found in the last section 4.

In the proof we use so-called the eigenvalues method, see e.g. [15] and some observations from [14].

2 Notation

Let \mathbf{G} be an abelian group and $+$ be the group operation. In the paper we use the same letter to denote a set $S \subseteq \mathbf{G}$ and its characteristic function $S : \mathbf{G} \rightarrow \{0, 1\}$. By $|S|$ denote the cardinality of S .

Let $f, g : \mathbf{G} \rightarrow \mathbb{C}$ be two functions with finite supports. Put

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y)g(y + x) \quad (1)$$

Let $A_1, \dots, A_k \subseteq \mathbf{G}$ be any sets. Put

$$\mathbf{E}_k(A_1, \dots, A_k) = \sum_{x \in \mathbf{G}} (A_1 \circ A_1)(x) \dots (A_k \circ A_k)(x) \quad (2)$$

be the higher energy of A_1, \dots, A_k . If $A_j = A$, $j = 1, \dots, k$ we simply write $E_k(A)$ instead of $E_k(A, \dots, A)$. In the same way one can define $E_k(A)$ for non-integer k . In particular case $k = 2$ we put $E(A, B) := E_2(A, B)$ and $E(A) = E_2(A)$. The quantity $E(A)$ is called the additive energy of a set, see e.g. [17]. Similarly, we define

$$E_k(f_1, \dots, f_k) = \sum_x (f_1 \circ f_1)(x) \dots (f_k \circ f_k)(x). \quad (3)$$

Denote by $\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k)$ the function

$$\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k) = \sum_z f_1(z) f_2(z + x_1) \dots f_{k+1}(z + x_k).$$

Thus, $\mathcal{C}_2(f_1, f_2)(x) = (f_1 \circ f_2)(x)$. If $f_1 = \dots = f_{k+1} = f$ then write $\mathcal{C}_{k+1}(f)(x_1, \dots, x_k)$ for $\mathcal{C}_{k+1}(f_1, \dots, f_{k+1})(x_1, \dots, x_k)$. Note that

$$\sum_{x_1, \dots, x_k} \mathcal{C}_{k+1}^2(f_1, \dots, f_{k+1})(x_1, \dots, x_k) = E_{k+1}(f_1, \dots, f_{k+1}). \quad (4)$$

Let $g : \mathbf{G} \rightarrow \mathbb{C}$ be a function, and $A \subseteq \mathbf{G}$ be a finite set. By T_A^g denote the matrix with indices in the set A

$$T_A^g(x, y) = g(x - y) A(x) A(y). \quad (5)$$

It is easy to see that T_A^g is hermitian iff $\overline{g(-x)} = g(x)$. The corresponding action of T_A^g is

$$\langle T_A^g a, b \rangle = \sum_z g(z) (\bar{b} \circ a)(z).$$

for any functions $a, b : A \rightarrow \mathbb{C}$. In the case $\overline{g(-x)} = g(x)$ by $\text{Spec}(T_A^g)$ we denote the spectrum of the operator T_A^g

$$\text{Spec}(T_A^g) = \{\mu_1 \geq \mu_2 \geq \dots \geq \mu_{|A|}\}. \quad (6)$$

Write $\{f\}_\alpha$, $\alpha \in [|A|]$ for the corresponding eigenfunctions. We call μ_1 as the main eigenvalue and f_1 as the main function.

In the asymmetric case let $g : \mathbf{G} \rightarrow \mathbb{C}$ be a function, and $A, B \subseteq \mathbf{G}$ be two finite sets. Suppose that $|B| \leq |A|$. By $T_{A,B}^g$ denote the rectangular matrix

$$T_{A,B}^g(x, y) = g(x - y) A(x) B(y), \quad (7)$$

and by $\tilde{T}_{A,B}^g(x, y)$ denote the another rectangular matrix

$$\tilde{T}_{A,B}^g(x, y) = g(x + y) A(x) B(y). \quad (8)$$

As in (6), we arrange the singular values in order of magnitude

$$\lambda_1(T_{A,B}^g) \geq \lambda_2(T_{A,B}^g) \geq \dots \geq \lambda_{|B|}(T_{A,B}^g) \geq 0,$$

$$T_{A,B}^g(x, y) = \sum_{j=1}^{|B|} \lambda_j u_j(x) v_j(y)$$

and similar for $\tilde{T}_{A,B}^g$. Here $u_j(x)$, $v_j(y)$ are singularfunctions of the operators. General theory of such operators was developed in [15].

All logarithms are base 2. Signs \ll and \gg are the usual Vinogradov's symbols.

3 The main definition

We begin with a rather general definition of families of sets which are usually obtained by Szemerédi–Trotter’s theorem, see [17].

Definition 5 A set $A \subset \mathbb{R}$ has **SzT–type** (in other words A is called **Szemerédi–Trotter set**) with parameter $\alpha \geq 1$ if for any set $B \subset \mathbb{R}$ and an arbitrary $\tau \geq 1$ one has

$$|\{x \in A + B : (A * B)(x) \geq \tau\}| \ll c(A)|B|^\alpha \cdot \tau^{-3}, \quad (9)$$

where $c(A) > 0$ is a constant depends on the set A only. We define the quantity $c(A)|B|^\alpha$ as $c(A, B)$.

From the definition one can see that if A has SzT–type then $(-A)$ has the same SzT–type with the same parameters α and $c(A)$.

Remark 6 We put parameter $\alpha \geq 1$ because otherwise there is no any SzT–type set. Indeed, take $B = C - A$, where set C will be chosen later. One has $(A * B)(x) \geq |A|$ for any $x \in C$. Then by (9), we obtain

$$|A|^3|C| \ll c(A)|A + C|^\alpha \leq c(A)|A|^\alpha|C|^\alpha.$$

Taking $|C|$ sufficiently large and having the set A is fixed, we see that $\alpha \geq 1$.

Examples. Let us give some examples of SzT–type sets with parameter $\alpha = 2$.

- 1) If $A \subset \mathbb{R}$ is a convex set then A has SzT–type with $c(A) = |A|$, see [13].
- 2) Let f be a strictly convex/concave function. Then $f(A)$ has SzT–type with $c(A) = q(A)$, where

$$q(A) := \min_C \frac{|A + C|^2}{|C|}, \quad (10)$$

and A has SzT–type with $c(A) = q(f(A))$, see [12], [9].

- 3) Let $|AA| \leq M|A|$. Then A has SzT–type with $c(A) = M^2|A|$. This is a particular case of the family from (2). Indeed, take $f(x) = \log x$, and apply (10) with $C = \log(A \cap \mathbb{R}^+)$ or $C = \log|(A \cap \mathbb{R}^-)|$.
- 4) Let $A \subset \mathbb{R}^+$, and $a \in \mathbb{R} \setminus \{0\}$. Then $\log A$ has SzT–type with $c(A) = q'(A)$, where

$$q'(A) := \min_C \frac{|(A + a)C|^2}{|C|},$$

see [6], [10].

There are another families of SzT–type sets, for example see a family of complex sets in [7].

Using definition 5 and easy calculations, one can obtain upper bounds for some simple characteristics of SzT–type sets see, e.g. papers [12], [13], [8], [9]. It is more convenient do not use parameter α in the statements.

Lemma 7 *Let A be a SzT-type set. Then*

$$E_3(A) \ll c(A, A) \log |A|, \quad E^3(A) \ll E_{3/2}^2(A) c(A, A),$$

and for any B one has

$$E(A, B) \ll (c(A, B) |A| |B|)^{1/2}.$$

We need in one more technical lemma.

Lemma 8 *Let A, A_* be a SzT-type set with the same parameter $\alpha > 1$. Then*

$$E^{2\alpha-1}(A_*, A) \ll_{(\alpha-1)^{-1}} \left(\sum_x (A_* \circ A_*)^{1/2}(x) (A \circ A)(x) \right)^{2\alpha-2} \cdot c^{1/3}(A) c^{\alpha/3}(A_*) |A|^{2/3} |A_*|^{\alpha^2/3}.$$

Proof. Put $c_* = c(A_*)$, $c = c(A)$, $a = |A|$, $a_* = |A_*|$. Splitting the sum, we get with some inaccuracy

$$E(A_*, A) \ll \tau^{1/2} \left(\sum_x (A_* \circ A_*)^{1/2}(x) (A \circ A)(x) \right) + \tau \sum_x S_\tau(x) (A \circ A)(x) = \tau^{1/2} \omega_1 + \tau \omega_2, \quad (11)$$

where $S_\tau = \{x : (A_* \circ A_*)(x) \geq \tau\}$. Because of A_* is Szemerédi–Trotter set, we have $|S_\tau| \ll c_*(a_*)^\alpha \tau^{-3}$. On the other hand, A is also Szemerédi–Trotter set, so

$$\sum_x S_\tau(x) (A \circ A)(x) = \sum_{x \in A} (S_\tau * A)(x) \ll c^{1/3} |S_\tau|^{\alpha/3} a^{2/3} \ll c^{1/3} a^{2/3} c_*^{\alpha/3} (a_*)^{\alpha^2/3} \tau^{-\alpha}. \quad (12)$$

Combining (11) and (12), we obtain

$$E(A_*, A) \ll_{(\alpha-1)^{-1}} \tau^{1/2} \omega_1 + \tau^{1-\alpha} c^{1/3} a^{2/3} c_*^{\alpha/3} (a_*)^{\alpha^2/3}.$$

An optimal choice of parameter τ is $\tau^{1/2} = \omega_1^{-1/(2\alpha-1)} c^{1/3(2\alpha-1)} c_*^{\alpha/3(2\alpha-1)} a^{2/3(2\alpha-1)} a_*^{\alpha^2/3(2\alpha-1)}$. Thus

$$E^{2\alpha-1}(A_*, A) \ll_{(\alpha-1)^{-1}} \left(\sum_x (A_* \circ A_*)^{1/2}(x) (A \circ A)(x) \right)^{2\alpha-2} \cdot c^{1/3} c_*^{\alpha/3} a^{2/3} a_*^{\alpha^2/3}$$

as required. \square

4 The proof of the main result

We begin with a lemma from [15].

Lemma 9 *Let $A \subseteq \mathbf{G}$ be a set and g be a nonnegative function on \mathbf{G} . Suppose that f_1 is the main eigenfunction of T_A^g or \tilde{T}_A^g , and μ_1 is the correspondent eigenvalue. Then*

$$\langle T_A^{A \circ A} f_1 f_1 \rangle \geq \frac{\mu_1^3}{\|g\|_2^2 \cdot \|g\|_\infty}.$$

A particular case $\alpha = 2$ of the next lemma is contained inside Theorem 8 of paper [14]. We give the proof for the sake of completeness.

Lemma 10 *Let A be a SzT-type set and let $\Delta \geq 1$ be a real number. Suppose that*

$$B \subseteq \{x : (A \circ A)(x) \geq \Delta\} \quad \text{or} \quad B \subseteq \{x : (A * A)(x) \geq \Delta\}.$$

Then

$$E_3(A, A, B) \ll \Delta^{-\frac{4}{3\alpha-1}} c(A)^{\frac{5\alpha+1}{2(3\alpha-1)}} |A|^{\frac{2\alpha^2+5\alpha-1}{2(3\alpha-1)}} |B|^{\frac{3(\alpha^2-1)}{2(3\alpha-1)}} \log |A|. \quad (13)$$

Proof. Put $a = |A|$, $L = \log a$ and $c = c(A)$. By the pigeonhole principle there is a set Q such that

$$\sigma := E_3(A, A, B) \ll L \sum_{x \in Q} (A \circ A)^2(x) (B \circ B)(x),$$

and the values of the convolution $(A \circ A)(x)$ differ at most twice on Q . Denote by q the maximum of $(A \circ A)(x)$ on Q . Because of A is SzT-type set, we have $|Q| \ll ca^\alpha q^{-3}$. Using Lemma 7, we obtain

$$\sigma \ll LqE(A, B) \ll Lq(c(A)a|B|^{\alpha+1})^{1/2}. \quad (14)$$

On the other hand, by the definition of the set B and the Cauchy-Schwarz inequality one has

$$\sigma \ll Lq^2 \Delta^{-1} E(Q, B, A, A) \ll Lq^2 \Delta^{-1} E^{1/2}(Q, A) E^{1/2}(B, A). \quad (15)$$

Combining the last two bounds and the upper bound for size of Q , we have

$$\begin{aligned} \sigma &\ll LqE^{1/2}(A, B)(E^{1/2}(A, B) + q\Delta^{-1}c^{1/4}a^{1/4}|Q|^{(\alpha+1)/4}) \ll \\ &\ll LqE^{1/2}(A, B)(E^{1/2}(A, B) + q^{-(3\alpha-1)/4}\Delta^{-1}c^{(\alpha+2)/4}a^{(\alpha^2+\alpha+1)/4}). \end{aligned}$$

The optimal choice of q is $q = E^{-2/(3\alpha-1)}(A, B)\Delta^{-4/(3\alpha-1)}c^{(\alpha+2)/(3\alpha-1)}a^{(\alpha^2+\alpha+1)/(3\alpha-1)}$. Here we have used the fact that $\alpha > 1/3$. Substituting q into the last formula and using Lemma 7 again, we get

$$\begin{aligned} \sigma &\ll LqE(A, B) \ll LE^{3(\alpha-1)/(3\alpha-1)}(A, B) \cdot \Delta^{-4/(3\alpha-1)}c^{(\alpha+2)/(3\alpha-1)}a^{(\alpha^2+\alpha+1)/(3\alpha-1)} = \\ &= L\Delta^{-\frac{4}{3\alpha-1}}|B|^{\frac{3(\alpha^2-1)}{6\alpha-2}}c^{\frac{5\alpha+1}{6\alpha-2}}a^{\frac{2\alpha^2+5\alpha-1}{6\alpha-2}} \end{aligned}$$

as required. \square

Let us formulate the main result of the paper.

Theorem 11 *Suppose that $A \subset \mathbb{R}$ has SzT-type with parameter α . Then*

$$|A + A| \gg c(A)^{\frac{1-11\alpha}{3\alpha^2+12\alpha+1}} |A|^{\frac{-8\alpha^2+57\alpha-3}{3\alpha^2+12\alpha+1}} \cdot (\log |A|)^{-\frac{4(3\alpha-1)}{3\alpha^2+12\alpha+1}}. \quad (16)$$

In particular, for $\alpha = 2$ one has

$$|A + A| \gg c(A)^{-\frac{21}{37}} |A|^{\frac{79}{37}} \cdot (\log |A|)^{-\frac{20}{37}}. \quad (17)$$

Proof. Let $S = A + A$, $|S| = d$, $a = |A|$. Let also $L = \log a$, $c = c(A)$. We have

$$|A|^2 = \sum_{x,y} A(x)A(y)S(x+y) \leq 2 \sum_{z \in S_1} (A * A)(z), \quad (18)$$

where $S_1 = \{z \in S : (A * A)(z) \geq 2^{-1}a^2d^{-1}\}$. Denote by f_j , μ_j the eigenfunctions and eigenvalues of hermitian operator $\tilde{T}_A^{S_1}$. From (18) it follows that $\mu_1 \geq a/2$. Applying Lemma 9, we see that

$$\langle T_A^{A \circ A} f_1, f_1 \rangle \geq \mu_1^3 (\tilde{T}_A^{S_1}) d^{-1} \geq 2^{-3} a^3 d^{-1}. \quad (19)$$

Further, by nonnegativity of the operator $T_A^{A \circ A}$ as well as inequality (19) and the lower bound for μ_1 , we get

$$\sigma := \sum_{x,y,z} T_A^{A \circ A}(x,y) \tilde{T}_A^{S_1}(x,z) \tilde{T}_A^{S_1}(y,z) = \sum_{j=1}^a \mu_j^2 \langle T_A^{A \circ A} f_j, f_j \rangle \geq \mu_1^2 \langle T_A^{A \circ A} f_1, f_1 \rangle \geq 2^{-5} a^5 d^{-1}. \quad (20)$$

On the other hand

$$\sigma = \sum_{x,y,z \in A} (A \circ A)(x-y) S_1(x+z) S_1(y+z) = \sum_{\alpha, \beta} S_1(\alpha) S_1(\beta) (A \circ A)(\alpha - \beta) \mathcal{C}_3(-A, A, A)(\alpha, \beta). \quad (21)$$

Combining (20), (21) and using (4), we obtain by the Cauchy–Schwarz inequality that

$$a^{10} d^{-2} \ll E_3(A) E_3(A, A, S_1). \quad (22)$$

Applying the first formula of Lemma 7 to estimate the quantity $E_3(A)$ and Lemma 10 to estimate $E_3(A, A, S_1)$, we have

$$a^{10} d^{-2} \ll L^2 a^\alpha c \cdot \Delta^{-\frac{4}{3\alpha-1}} c^{\frac{5\alpha+1}{2(3\alpha-1)}} a^{\frac{2\alpha^2+5\alpha-1}{2(3\alpha-1)}} d^{\frac{3(\alpha^2-1)}{2(3\alpha-1)}},$$

where $\Delta = 2^{-1}a^2d^{-1}$. After some calculations, we get

$$d \gg L^{-\frac{4(3\alpha-1)}{3\alpha^2+12\alpha+1}} c^{\frac{1-11\alpha}{3\alpha^2+12\alpha+1}} a^{\frac{-8\alpha^2+57\alpha-3}{3\alpha^2+12\alpha+1}}$$

as required. \square

Proof of Theorems 2, 4. To obtain Theorem 2 just recall that $\alpha = 2$, $c(A) = |A|$ for convex sets. Remembering the definition of $q(f(A))$ from (10), we have $c(A) = q(f(A)) \leq |f(A) + C|^2 |C|^{-1}$. After that applying the main Theorem 11, we get Theorem 4. \square

Remark 12 Of course, one can replace the condition $|A| = |C|$ in Theorems 3, 4 to $c_1|A| \leq |C| \leq c_2|A|$, where $c_1, c_2 > 0$ are any absolute constants. Certainly, signs \ll, \gg should be changed by $\ll_{c_1, c_2}, \gg_{c_1, c_2}$ in the case. Even more, it is possible to prove the results for sets A and C , having incomparable sizes. We do not make such calculations here (and also below), note only that because in Theorem 4, we have $c(A) \leq |f(A) + C|^2|C|^{-1}$ this implies

$$|f(A) + C|^{42}|A + A|^{37} \gg |A|^{79}|C|^{21} \log^{-20}|A|.$$

We conclude the section proving a result which generalize, in particular, Theorem 1.3 from [9] as well as the results on sumsets/difference sets of convex sets from [13]. The arguments are in the spirit of Theorem 11. We need in a lemma from [15].

Lemma 13 Let $A, B \subseteq \mathbf{G}$ be finite sets, $|B| \leq |A|$, $D, S \subseteq \mathbf{G}$ be two sets such that $A - B \subseteq D$, $A + B \subseteq S$. Then the main eigenvalues and singularfunctions of the operators $T_{A,B}^D, \tilde{T}_{A,B}^S$ equal $\lambda_1 = (|A||B|)^{1/2}$, and

$$v_1(y) = B(y)/|B|^{1/2}, \quad \text{and} \quad u_1(x) = A(x)/|A|^{1/2}.$$

All other singular values equal zero.

Using lemma above, we prove our second main result, although one can use a more elementary approach as in [13].

Theorem 14 Suppose that $A, A_* \subset \mathbb{R}$ have SzT-type with the same parameter α . Then

$$\begin{aligned} & |A \pm A_*| \gg \\ & \min\{c(A_*)^{-\frac{2}{3(7+\alpha)}} c(A)^{-\frac{13}{3(7+\alpha)}} |A_*|^{\frac{2(24-\alpha)}{3(7+\alpha)}} |A|^{\frac{33-10\alpha}{3(7+\alpha)}}, c(A)^{-\frac{2}{3(7+\alpha)}} c(A_*)^{-\frac{13}{3(7+\alpha)}} |A|^{\frac{2(24-\alpha)}{3(7+\alpha)}} |A_*|^{\frac{33-10\alpha}{3(7+\alpha)}}\} \\ & \times (\log(|A||A_*|))^{-\frac{2}{7+\alpha}}, \end{aligned} \quad (23)$$

and for $\alpha > 1$

$$\begin{aligned} & |A \pm A_*| \gg_{(\alpha-1)^{-1}} c(A)^{-\frac{4\alpha-2}{3(\alpha^2+4\alpha-3)}} c(A_*)^{-\frac{7\alpha-5}{3(\alpha^2+4\alpha-3)}} |A|^{\frac{28\alpha-4\alpha^2-16}{3(\alpha^2+4\alpha-3)}} |A_*|^{\frac{35\alpha-4\alpha^2-21}{3(\alpha^2+4\alpha-3)}} \\ & \times (\log(|A||A_*|))^{-\frac{2(\alpha-1)}{\alpha^2+4\alpha-3}}. \end{aligned} \quad (24)$$

In particular, for $\alpha = 2$ one has

$$\begin{aligned} & |A \pm A_*| \gg \max\{c(A_*)^{-\frac{1}{3}} c(A)^{-\frac{2}{9}} |A_*|^{\frac{11}{9}} |A|^{\frac{8}{9}}, c(A)^{-\frac{1}{3}} c(A_*)^{-\frac{2}{9}} |A|^{\frac{11}{9}} |A_*|^{\frac{8}{9}}, \\ & \min\{c(A_*)^{-\frac{2}{27}} c(A)^{-\frac{13}{27}} |A_*|^{\frac{44}{27}} |A|^{\frac{13}{27}}, c(A)^{-\frac{2}{27}} c(A_*)^{-\frac{13}{27}} |A|^{\frac{44}{27}} |A_*|^{\frac{13}{27}}\} \} \times \\ & \times (\log(|A||A_*|))^{-\frac{2}{9}}. \end{aligned} \quad (25)$$

Finally

$$|A \pm A_*|^{\frac{\alpha+1}{2}} |A - A| \gg |A|^{\frac{33-4\alpha}{6}} |A_*|^{\frac{6-\alpha}{3}} c^{-7/6}(A) c^{-1/3}(A_*) \log^{-1}(|A||A_*|). \quad (26)$$

Proof. Because SzT-types of the sets A and $(-A)$ are coincide it is sufficient to prove the result for sums. Let $S = A + A_*$, $|S| = d$, $a = |A|$, $a_* = |A_*|$. Let also $L = \log(aa_*)$, $c = c(A)$, $c_* = c(A_*)$. By the Cauchy–Schwarz inequality, we have

$$aa_* = \sum_{x,y} A(x)A_*(y)S(x+y) \leq d^{1/2}E^{1/4}(A)E^{1/4}(A_*). \quad (27)$$

Let us begin with (23). One can assume that $E(A) \geq d^{-1}a^2a_*^2$, the opposite case is similar. By Lemma 7, we get

$$\frac{E_{3/2}(A)}{a} \geq d^{-3/2}c^{-1/2}a^{2-\alpha/2}a_*^3. \quad (28)$$

Denote by f_j , μ_j the eigenfunctions and the eigenvalues of hermitian nonnegative operator

$$(\tilde{T}_{A,A_*}^S(\tilde{T}_{A,A_*}^S)^*)(x,y) = \mathcal{C}_3(A_*, S, S)(x,y)A(x)A(y).$$

By Lemma 13 we know that $f_1(x) = A(x)/a^{1/2}$ and $\mu_1 = aa_*$. Using the lemma again as well as bound (28), we obtain

$$\begin{aligned} \sigma &:= \sum_{x,y \in A} T_A^{(A \circ A)^{1/2}}(x,y) \mathcal{C}_3(A_*, S, S)(x,y) = \sum_{j=1}^a \mu_j \langle T_A^{(A \circ A)^{1/2}} f_j, f_j \rangle = \\ &= a^{-1} \mu_1 \langle T_A^{(A \circ A)^{1/2}} A, A \rangle \geq d^{-3/2}c^{-1/2}a^{3-\alpha/2}a_*^4. \end{aligned} \quad (29)$$

On the other hand, we have as in (21), (22) that

$$\sigma^2 \leq E_3(A_*, A, A)E(A, S) \leq E_3(A_*, A, A)(cad^{\alpha+1})^{1/2}. \quad (30)$$

Using calculations similar to Lemma 7, one can show that

$$E_3(A_*, A, A) \ll (c_*c^2)^{1/3}(a_*a^2)^{\alpha/3}L. \quad (31)$$

Substituting (31) into (30) and combining the result with (29), we obtain

$$d^{-3}c^{-1}a^{6-\alpha}a_*^8 \leq (cad^{\alpha+1})^{1/2}(c_*c^2)^{1/3}(a_*a^2)^{\alpha/3}L.$$

After some calculations, we have

$$d \gg L^{-\frac{2}{7+\alpha}}c_*^{-\frac{2}{3(7+\alpha)}}c^{-\frac{13}{3(7+\alpha)}}a_*^{\frac{2(24-\alpha)}{3(7+\alpha)}}a^{\frac{33-10\alpha}{3(7+\alpha)}}$$

as required.

To prove (24), returning to (27) and applying Lemma 8, we obtain

$$(a^2a_*^2d^{-1})^{2\alpha-1} \leq E^{2\alpha-1}(A_*, A) \ll_{(\alpha-1)^{-1}} \left(\sum_x (A_* \circ A_*)^{1/2}(x)(A \circ A)(x) \right)^{2\alpha-2} \cdot c^{1/3}c_*^{\alpha/3}a^{2/3}a_*^{\alpha^2/3}.$$

Thus

$$a^{-1} \langle T_A^{(A_* \circ A_*)^{1/2}} A, A \rangle \gg_{(\alpha-1)^{-1}} a^{\frac{3\alpha-1}{3(\alpha-1)}}a_*^{\frac{12\alpha-6-\alpha^2}{6(\alpha-1)}}d^{\frac{1-2\alpha}{2(\alpha-1)}}c^{-\frac{1}{6(\alpha-1)}}c_*^{-\frac{\alpha}{6(\alpha-1)}}. \quad (32)$$

After that use previous arguments replacing $T_A^{(A \circ A)^{1/2}}$ onto $T_A^{(A_* \circ A_*)^{1/2}}$. By (32), we have

$$\begin{aligned} a^{\frac{12\alpha-8}{3(\alpha-1)}} a_*^{\frac{18\alpha-12-\alpha^2}{3(\alpha-1)}} d^{\frac{1-2\alpha}{\alpha-1}} c^{-\frac{1}{3(\alpha-1)}} c_*^{-\frac{\alpha}{3(\alpha-1)}} &\leq \left(\mu_1 a^{-1} \langle T_A^{(A_* \circ A_*)^{1/2}} A, A \rangle \right)^2 \leq E_3(A_*, A, A) E(A_*, S) \leq \\ &\leq (c_* a_* d^{\alpha+1})^{1/2} (c_* c^2)^{1/3} (a_* a^2)^{\alpha/3} L. \end{aligned}$$

After some calculations, we get the required bound.

Finally, to get (26) just apply the arguments above to get

$$a_*^2 E_{3/2}^2 \leq (cad^{\alpha+1})^{1/2} (c_* c^2)^{1/3} (a_* a^2)^{\alpha/3} L$$

and use the lower bound $E_{3/2}^2(A) \geq a^6/|A-A|$. Of course, one can replace A to A_* in (26) and vice versa. This completes the proof. \square

Theorem above gives us a consequence to sumsets/difference sets for convex sets.

Corollary 15 *Let $A, A_* \subset \mathbb{R}$ be two convex sets. Then*

$$|A \pm A_*| \gg \max\{|A|^{\frac{8}{9}} |A_*|^{\frac{2}{3}}, |A_*|^{\frac{8}{9}} |A|^{\frac{2}{3}}\} \cdot \log^{-\frac{2}{9}}(|A||A_*|).$$

In another corollary we obtain Theorem 1.3 from [9] as well as Corollary 1.4 from the paper. These results can be considered as theorems on lower bounds for sums of SzT-type sets of special form.

Corollary 16 *We have*

$$|A + f(A)| \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}} \quad (33)$$

for any strictly convex or concave function f . Further

$$|AA|^6 |A - A|^5 \gg \frac{|A|^{14}}{\log^2 |A|}. \quad (34)$$

In particular

$$\max\{|AA|, |A - A|\} \gg |A|^{14/11} \log^{-2/11} |A|.$$

Proof. Indeed, to obtain (33) just apply (25) with $A = A$, $A_* = f(A)$ and $c(A) = c_*(A) = |f(A) + A|^2 |A|^{-1}$. To get (34), we use (26) with $A = A_*$ having

$$|A - A|^{5/2} \gg |A|^{11/2} c^{-3/2}(A) \log^{-1} |A|.$$

After that recall $c(A) = M^2 |A|$ with $M = |AA|/|A|$. This concludes the proof. \square

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